

A Note on Topological Amenability

Nicolas Monod

École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland

Correspondence to be sent to: nicolas.monod@epfl.ch

A simple characterization of topological amenability in terms of bounded cohomology is proved, following Johnson's formulation of amenability. The connection to injective Banach modules is established.

1 Introduction

1.1 Motivation

According to Johnson–Ringrose [18, Section 2], a locally compact group G is amenable if and only if the (continuous) bounded cohomology $H_b^n(G, E^*)$ vanishes for all $n \geq 1$ and all (continuous) Banach G -modules E . Here, E^* denotes the dual module; the very nature of amenability as a limiting property explains the use of duals and weak limits. The corresponding condition on *all* modules simply characterizes compact groups instead. In Johnson's terminology, $H_b^n(G, -)$ is the cohomology of the Banach algebra L^1G . The importance of this reformulation is that it led to Johnson's definition of amenable algebras.

The purpose of this note is to establish the corresponding characterization of *amenable actions* in the topological sense. One motivation is the particular case of *exactness* of discrete groups.

Received May 28, 2010; Revised September 26, 2010; Accepted October 01, 2010
Communicated by Prof. Marc Burger

Definitions will be recalled in the text as needed. For now, keeping in mind the above remark on duals and limiting phenomena, we just propose the slogan that amenable actions are to proper actions what amenable groups are to compact groups.

This note grew out of my reading of [9] where a partial result for the case of exactness is obtained by another approach. I have been informed by Brodzki, Niblo, Nowak, and Wright that they have previously obtained a characterization of amenable actions. After completing this note I received their preprint containing similar ideas (now available as [2]); the reader is encouraged to consult that article as well.

1.2 Topological amenability

Let G be a group acting on the compact space X . We always assume that X is Hausdorff and that the action is by homeomorphisms. There is no topology on G (although our discussion of relative injectivity applies to locally compact groups also).

What is desired is a cohomological characterization of the amenability of this action. It should extend the Johnson–Ringrose criterion for the amenability of (discrete) groups. We propose, therefore, to consider the vanishing of $H_b^n(G, E^*)$ for a class of Banach G -modules E determined by X . The condition imposed by X should be void when X is a point, and be functorial. Before going any further, there is an obvious guess: all modules of the form $E = C(X, V)$ where V is any Banach G -module. It turns out that there is indeed an easy criterion along those lines:

The G -action on X is topologically amenable if and only if $H_b^n(G, C(X, V)^{**})$ vanishes for every Banach G -module V and every $n \geq 1$.

An aesthetic drawback is that we passed here to the bidual E^{**} , whilst one expects a criterion involving all duals; furthermore, we would like more general and more intrinsic classes of modules than $E = C(X, V)$. We shall, therefore, move beyond this obvious first guess and return to general Banach G -modules E . The influence of X will simply be through the data of a $C(X)$ -module structure, which should of course be compatible with the G -action.

Let us therefore call E a (G, X) -module if it is both a Banach G - and $C(X)$ -module, such that $g\varphi g^{-1}$ corresponds to the action of $g \in G$ on $\varphi \in C(X)$. This purely algebraic structure is too general to capture the analytic situation: Example 8 illustrates that it *cannot* characterize amenability. This reflects the fact that not all (G, X) -modules are modules for the underlying crossed product algebra introduced below. Analyzing only the $C(X)$ -structure alone, we can, however, isolate a relevant concept inspired by Kakutani’s classical work [19, 20]:

Definition 1. We say that a $C(X)$ -module E is *of type M* if for all $u \in E$ and $\varphi_i \in C(X)$ with $\varphi_i \geq 0$ one has

$$\|\varphi_1 u\| + \cdots + \|\varphi_n u\| \leq \|\varphi_1 + \cdots + \varphi_n\| \cdot \|u\|. \quad (\text{M})$$

We say that E is *of type C* if for all $u_i \in E$ and $\varphi_i \geq 0$ one has

$$\|\varphi_1 u_1 + \cdots + \varphi_n u_n\| \leq \|\varphi_1 + \cdots + \varphi_n\| \cdot \max_i \|u_i\|. \quad (\text{C})$$

□

(These notions are mutually dual, yet both types include modules that are not dual. The fundamental examples motivating the terminology are modules of measures, respectively, of continuous functions.)

We now have the criterion that we sought:

Proposition (Concise version). Let G be a group acting on the compact space X . The following are equivalent.

- (i) The G -action on X is topologically amenable.
- (ii) $H_b^1(G, E^*) = 0$ for every (G, X) -module E of type M.
- (iii) $H_b^n(G, E^*) = 0$ for every (G, X) -module E of type M and every $n \geq 1$. □

1.3 Injective modules and means

Our cohomological characterization extends to a further aspect not covered by Johnson–Ringrose, namely *injective modules* (see Section 2.1). By way of motivation, we recall the analogous situation in ergodic theory.

For measure spaces, amenable actions in Zimmer’s sense [27] are characterized by the *relative injectivity* of the corresponding modules of L^∞ -maps [6, 23]. We shall provide a similar statement for topological amenability. However, the notion of “bounded measurable maps” becomes problematic when X is non-metrizable, as it will definitely be for applications to exactness where X is the Čech–Stone space βG . We submit that a good analog is provided in complete generality by the dual of the space $\mathcal{I}(C(X), W)$ of *integral operators* valued in any Banach space W (in the sense of Grothendieck [13, Section 2] for abstract integral operators between arbitrary Banach spaces). Hence the statements (iii) and (vii). As it turns out, this criterion applies to all dual modules of type C, see (viii).

Finally, as in the Johnson–Ringrose criterion, we transit through a characterization of amenability in terms of invariant means. The equivalence of this with the definition of amenability is routine, but we derive it for completeness, see (ix) and (x). The summation condition therein means that the canonical image in $C(X)^{**}$ of the invariant element is the bidual of the constant function $1_X \in C(X)$.

Proposition (Full version). Let G be a group acting on the compact space X . The following are equivalent.

- (i) The G -action on X is topologically amenable.
- (ii) $H_b^n(G, C(X, V)^{**}) = 0$ for every Banach G -module V and every $n \geq 1$.
- (iii) $H_b^n(G, \mathcal{I}(C(X), W)^*) = 0$ for every Banach G -module W and every $n \geq 1$.
- (iv) $H_b^n(G, E^*) = 0$ for every (G, X) -module E of type M and every $n \geq 1$.
- (v) Any of the previous three points holds for $n = 1$.
- (vi) $C(X, V)^{**}$ is relatively injective for every Banach G -module V .
- (vii) $\mathcal{I}(C(X), W)^*$ is relatively injective for every Banach G -module W .
- (viii) Every dual (G, X) -module of type C is a relatively injective Banach G -module.
- (ix) There is a G -invariant element in $C(X, \ell^1 G)^{**}$ summing to 1_X .
- (x) There is a norm 1 positive G -invariant element in $C(X, \ell^1 G)^{**}$ summing to 1_X . □

Remarks. (a) The proof shows that it suffices in (ix) and (x) that the summation be any function that never vanishes. The norm condition in (x) is redundant.

(b) The only substantially new fact is the implication (i) \Rightarrow (viii), which still holds for G locally compact second countable with the definitions of [1, Section 2] and [23, Section 4.1]. □

2 Functional Analytic Context

This section presents more context than is needed for the proofs; we decided to follow a natural path rather than the shortest one. In particular, the discussion of crossed products can be ignored by the reader in a hurry. For definiteness, let us agree that Banach spaces and algebras are real. The few facts on operators and tensor products that we will use can all be found in Grothendieck [13, 14] (or in the gloss [8]).

2.1 G -modules and cohomology

We will use almost nothing from the theory of bounded cohomology; for background, we refer to [11, 17, 18, 23]. A *Banach G -module* is a Banach space with an isometric linear G -representation, viewed also as a (left) $\ell^1 G$ -module. Johnson [18] allows also uniformly bounded modules; although this would not change anything to our arguments, we keep the isometric setting because exact norms are important when applying bounded cohomology to the study of characteristic classes [5, 11]. The Banach G -module E is *relatively injective* if any G -morphism $\alpha : A \rightarrow E$ from a Banach G -submodule $A \subseteq B$ can be extended to $\beta : B \rightarrow E$ with $\|\beta\| = \|\alpha\|$ provided that there is a norm one projection $B \rightarrow A$ (as Banach spaces).

2.2 $C(X)$ -modules

Given a compact space X , we endow the algebra $C(X)$ with the uniform norm. A (Banach) $C(X)$ -module is an algebraic module E such that $\|\varphi u\|_E \leq \|\varphi\| \cdot \|u\|_E$ for $\varphi \in C(X)$, $u \in E$. For instance, consider the module of (strongly) continuous maps $E = C(X, V)$ to any Banach space V , with norm $\|\eta\|_E = \sup_{x \in X} \|\eta(x)\|_V$. If V is also a Banach G -module, then E becomes a Banach G -module for the diagonal action. Moreover, it is a (G, X) -module as defined in the introduction. Recalling our definition of the types C and M, one checks that the $C(X)$ -module $C(X, E)$ is of type C. More generally, if Y is a compact space with a continuous map $Y \rightarrow X$, we obtain naturally a $C(X)$ -module structure on $C(Y, E)$ and it is of type C.

Proposition 2. Let E be a $C(X)$ -module.

- (i) E^* is of type C if and only if E is of type M.
- (ii) E^* is of type M if and only if E is of type C.
- (iii) Both conditions C and M descend to submodules.
- (iv) Each inequality implies the corresponding inequality for arbitrary $\varphi_i \in C(X)$ upon replacing φ_i by $|\varphi_i|$ on the right-hand side. □

Proof. The definition of the dual norm implies readily that E^* is of type C (respectively, M) if E is of type M (resp. C). The converse statements follow by duality after embedding E into E^{**} and using point (iii), which is obvious. Point (iv) follows from decomposing each φ_i into positive and negative parts. ■

Corollary 3. For any Banach space W , the $C(X)$ -module $\mathcal{I}(C(X), W)$ is of type M. \square

Remark 4. If W happens to be a dual $W = V^*$, then $\mathcal{I}(C(X), W) \cong C(X, V)^*$; but in general $\mathcal{I}(C(X), W)$ is not dual. \square

Proof of Corollary 3. Recall that $\mathcal{I}(C(X), W)$ sits isometrically as a closed subspace in the integral bilinear forms $\mathcal{B}_I(C(X) \times W^*)$. The latter is the dual of the injective tensor product $C(X) \check{\otimes} W^*$ which identifies with $C(X, W^*)$. All this is compatible with the $C(X)$ -structures. In other words, we realized $\mathcal{I}(C(X), W)$ as a submodule of $C(X, W^*)^*$, which is of type M. \blacksquare

2.3 The algebra $C(X, \ell^1 G)$

An important particular case is $\mathcal{A} := C(X, \ell^1 G)$. There are canonical identifications

$$\mathcal{A} := C(X, \ell^1 G) \cong C(X) \check{\otimes} \ell^1(G) \cong \text{uc}_G[C(X)],$$

where $\text{uc}_G[-]$ denotes unconditionally summable sequences indexed by G . Note, in particular, that the *finite* sums of the form $\sum_g \varphi_g \otimes \delta_g$ with $\varphi_g \in C(X)$ are dense in \mathcal{A} . Moreover, the \mathcal{A} -norm of such an element is

$$\left\| \sum_g \varphi_g \otimes \delta_g \right\|_{\mathcal{A}} = \left\| \sum_g |\varphi_g| \right\|_{C(X)}.$$

The (G, X) -structure turns \mathcal{A} into a Banach algebra that we call the *Banach crossed product* of G and X , not to be confused with the C^* -algebraic crossed product. Explicitly, the product is $(\varphi \otimes \delta_g)(\psi \otimes \delta_h) = \varphi g \psi \otimes \delta_{gh}$ on elementary tensors with $g, h \in G$ and $\varphi, \psi \in C(X)$. The natural algebraic involution $\varphi \otimes \delta_g \mapsto g^{-1} \varphi \otimes \delta_{g^{-1}}$ does *not* in general extend to \mathcal{A} (Example 9), and there is indeed a fundamental difference between right and left \mathcal{A} -modules, as we shall see.

Proposition 5. Let G be a group acting on the compact space X and let E be a (G, X) -module. Let $\mathcal{A} = C(X, \ell^1 G)$.

- (i) If E is of type C, then it is naturally a left \mathcal{A} -module.
- (ii) If E is of type M, then it is naturally a right \mathcal{A} -module. \square

Example 6. The (G, X) -module $C(X) \hat{\otimes} \ell^1 G$, where $\hat{\otimes}$ is the projective tensor product, is not an \mathcal{A} -module unless either G or X is finite. This is a special case of the Dvoretzky–Rogers Theorem [10] since $C(X) \hat{\otimes} \ell^1 G \cong \ell^1(G, C(X))$. In particular, it is not of type C; in addition, it is not of type M either unless X is a point. \square

Proof of Proposition 5. The condition (C) precisely shows that for finite sums $\alpha = \sum_g \alpha_g \otimes \delta_g$ and $u \in E$ we have

$$\|\alpha u\|_E = \left\| \sum_g \alpha_g(gu) \right\| \leq \left\| \sum_g |\alpha_g| \right\|_{C(X)} \cdot \max_g \|gu\|_E = \|\alpha\|_{\mathcal{A}} \cdot \|u\|_E$$

and thus the left multiplication extends to \mathcal{A} . If E is of type M, we define

$$u\alpha := \sum_g ((g^{-1}\alpha_g) \otimes \delta_{g^{-1}})u = \sum_g g^{-1}(\alpha_g u),$$

which is a right module structure over finite sums. We then carry out a similar computation using the inequality (M). \blacksquare

3 Amenable Actions

The action of a group G on a compact space X is called *topologically amenable* if there is a net $\{\mu^j\}_{j \in J}$ in $C(X, \ell^1 G)$ such that every $\mu^j(x)$ is a probability measure on G and

$$\lim_{j \in J} \|g\mu^j - \mu^j\|_{C(X, \ell^1 G)} = 0 \quad (\forall g \in G).$$

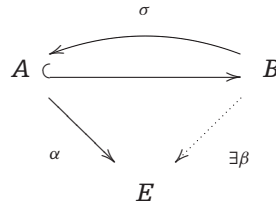
Note that if G is countable, the net can be replaced by a sequence; this does not require X to be metrizable and hence applies for instance to the Čech–Stone compactification of a countable group. Note also that the continuity of $\mu^j : X \rightarrow \ell^1 G$ can be replaced by weak-* continuity since both topologies coincide on probability measures. By compactness of X , one can assume that the support of $\mu^j(x)$ is in a finite set depending only on j (cf. Prokhorov’s criterion). Finally, we observe that, as expected, amenability is an “approximate properness” with μ^j being an approximate *Bruhat function*.

Background references are [1, 4]; we recall that the amenability of the action is equivalent to the nuclearity of the C^* -algebraic reduced crossed product, and thus in turn to its amenability, see Connes [7] and Haagerup [15].

3.1 Proofs

In order to take advantage of some simple implications, the proof will not follow a single cycle of implications. The heart of the matter is the following implication, which still holds for G locally compact second countable with the definitions of [1, Section 2; 23, Section 4.1].

(i) \Rightarrow (viii). Let E be a (G, X) -module of type C and consider the extension problem of Banach G -modules



where σ is a norm 1 projection which need not respect the module structure. (To be perfectly clear: the type C hypothesis regards only the $C(X)$ -structure, whilst the extension problem deals only with the G -structure.) Let $\mu \in C(X, \ell^1 G)$ be a finite sum $\sum_g \mu_g \otimes \delta_g$. We then define β_μ on each $b \in B$ by

$$\beta_\mu(b) := \sum_g (\mu_g \otimes \delta_g) \alpha(\sigma(g^{-1}b)).$$

In particular, β_μ is linear and $\beta_\mu|_A = \alpha$ if $\sum_g \mu_g = 1_X$. Moreover, the condition C implies $\|\beta_\mu\| \leq \|\mu\| \cdot \|\alpha\|$. If now E is a dual $E = V^*$, then those conditions are closed in the weak-* topology on $\mathcal{L}(B, E) \cong (B \hat{\otimes} V)^*$. Therefore, taking β to be a weak-* accumulation point of the family β_μ where μ ranges over a net as in the definition of topological amenability, it remains only to show that β is G -equivariant. For $h \in G$ and $b \in B$, we compute

$$\beta_\mu(hb) - h\beta_\mu(b) = \sum_g (\mu_g - h\mu_{h^{-1}g}) g \alpha(\sigma(g^{-1}hb))$$

which by condition (C) is bounded in E -norm by

$$\|\mu - h\mu\|_{C(X, \ell^1 G)} \cdot \|\alpha\| \cdot \|b\|_B.$$

It follows indeed that β is equivariant.

(viii) \Rightarrow (iv) and (vi) \Rightarrow (ii). Both are due to the fact that cohomology with values in relatively injective modules vanishes in every positive degree (e.g., because the module is a trivial resolution of itself). For (iv) we recall that E^* is of type C.

(viii) \Rightarrow (vii) \Rightarrow (vi) and (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (v) are immediate in view of Corollary 3 and Remark 4.

(v) \Rightarrow (ix). Consider the exact sequence of Banach G -modules

$$0 \longrightarrow C(X, \ell_0^1 G) \longrightarrow C(X, \ell^1 G) \longrightarrow C(X) \longrightarrow 0,$$

where $\ell_0^1 G$ is the kernel of the summation map (as a Banach space sequence it is even split exact). Apply the bounded cohomology long exact sequence to the bidual sequence and use the vanishing assumption for $C(X, \ell_0^1 G)^{**}$. This yields a short exact sequence of G -invariants

$$0 \longrightarrow (C(X, \ell_0^1 G)^{**})^G \longrightarrow (C(X, \ell^1 G)^{**})^G \longrightarrow (C(X)^{**})^G \longrightarrow 0$$

and (ix) follows.

(ix) \Rightarrow (i). Let $\{\mu^j\}_{j \in J}$ be a net in $C(X, \ell^1 G)$ converging to an element as in (ix) for the weak-* topology of the bidual. We can assume that each μ^j ranges in the functions in $\ell^1 G$ that sum up to one. Since $g\mu^j - \mu^j$ converges weakly to zero for all $g \in G$, a standard Hahn–Banach argument shows that, upon passing to a net of finite convex combinations, we can assume that $\|g\mu^j - \mu^j\|$ converges to zero for all g . Recall that $C(X, \ell^1 G)$ is a Banach lattice and that therefore all lattice operations are (norm-) continuous (see, e.g., [22, Vol. II] for Banach lattices). Moreover, they are G -equivariant. Recall further that canonical embeddings into biduals are compatible with all Banach lattice structures. In conclusion, replacing μ^j by its absolute value, we can further assume that all μ^j are nonnegative, retaining all properties except that now the image $\Sigma\mu^j := \sum_G \mu^j$ of μ^j in $C(X)$ is only *bounded below* by 1_X . However, the (equivariant) renormalization $\mu^j \mapsto \mu^j / \Sigma\mu^j$ is 2-Lipschitz on maps with $\Sigma\mu^j \geq 1_X$ and therefore we finally obtain a net as required.

We could replace 1_X in the assumption by any never vanishing $f \in C(X)$; the Lipschitz constant would then be $2/\min_x |f(x)|$.

(i) \Rightarrow (x) is obtained by choosing a weak-* accumulation point and keeping in mind the properties of Banach lattices recalled above. Since (x) \Rightarrow (ix) is tautological, this completes the proof.

4 Exact Groups and Further Remarks

The definition of exact groups goes back to Kirchberg–Wassermann [21]; it is equivalent to the statement that the reduced C^* -algebra of the group is exact for spatial tensor products. Ozawa [26] and Anantharaman-Delaroche [1] proved that the group G is exact if and only if its action on the Čech–Stone compactification βG is amenable. Higson–Roe [16] established that the latter condition is equivalent to Yu’s *Property A*. Exactness has deep implications for the group but at the same time it is notoriously difficult to produce any nonexact group; Gromov succeeded in [12] using the relation to uniform embeddings into Hilbert spaces.

In view of the equivalence with amenability of the action on βG , the propositions of the introduction characterize exact groups. In this special case, $C(\beta G) \cong \ell^\infty G$, allowing some additional identifications. For instance, $(G, \beta G)$ -modules are particularly concrete and can be thought of as modules over the involutive Banach–Hopf algebra $\ell^1 G$.

Example 7. Let V be a Banach G -module. Any G -invariant $\ell^\infty G$ -submodule of $\ell^\infty(G, V)$ is a $(G, \beta G)$ -module of type C. It is generally not of the form $C(\beta G, V)$. \square

This example provides a link to the work of Douglas–Nowak [9]. Another article investigating cohomological characterizations of exactness is [3].

Another obvious example of $(G, \beta G)$ -module is provided by the various $\ell^p G$.

Example 8. Let G be a non-Abelian free group. Then the G -action on $X := \beta G$ is amenable. On the other hand, $H_b^2(G, \ell^1 G)$ is non-trivial, even though $\ell^1 G$ is a dual (G, X) -module. In fact, it is of type M and is the dual of the (G, X) -module $c_0(G)$ of type C. More generally, $H_b^2(G, \ell^p G)$ is nontrivial for all $p < \infty$ using the argument of [25]. \square

The algebra

$$\mathcal{A} := C(\beta G, \ell^1 G) \cong \ell^\infty G \check{\otimes} \ell^1 G \cong \text{uc}_G[\ell^\infty G],$$

which coincides with the “uniform algebra” $\ell_u G$ of [9], also admits additional realizations since now any element can be viewed as a kernel on $G \times G$. We find thus:

$$\mathcal{A} \cong C(\ell^1 G) \cong \mathcal{L}_{w^*/w}(\ell^\infty G),$$

where \mathcal{C} denotes compact operators and $\mathcal{L}_{w^*/w}$ the weak- $*$ -to-weak continuous operators (which are necessarily compact), compare pp. 90–91 in Grothendieck [14]. (The product, meanwhile, remains G -twisted).

Example 9. The involution familiar from C^* -crossed products is not \mathcal{A} -bounded: if $S \subseteq G$ is a finite set, the diagonal $\sum_{g \in S} \delta_g \otimes \delta_g$ has unit norm but its image $\delta_e \otimes \sum_{g \in S} \delta_{g^{-1}}$ has norm $|S|$. \square

Returning to G -actions on general compact spaces X , we conclude with suggestions for further investigation.

Question 10. Let E be a (G, X) -module. In the spirit of Kakutani [19, 20], give a concrete expression of the intrinsic properties C and M . \square

Questions 11. (a) Characterize the amenability of $\mathcal{A} := C(X, \ell^1 G)$.

(b) Characterize (and define!) the co-amenability of the subalgebra $\ell^1 G$ in $\mathcal{A} := C(X, \ell^1 G)$, embedded by tensoring with 1_X . A natural candidate for the definition of co-amenable subalgebras is: The inclusion morphism induces an injective restriction morphism at the level of Banach algebra cohomology, for all dual modules over the target. (This is the faithful translation of Proposition 3 in [24].) \square

Funding

Supported in part by the Swiss National Science Foundation 200020-126817 and CRSI22-130435.

References

- [1] Anantharaman-Delaroche, C. "Amenability and exactness for dynamical systems and their C^* -algebras." *Transactions of the American Mathematical Society* 354, no.10 (2002): 4153–78.
- [2] Brodzki, J., G. A. Niblo, P. Nowak, and N. J. Wright. "Amenable actions, invariant means and bounded cohomology," (2010): preprint arXiv:1004.0295.
- [3] Brodzki, J., G. A. Niblo, and N. J. Wright. "A cohomological characterisation of Yu's Property A for metric spaces," (2010): preprint arXiv:1002.5040.
- [4] Brown, N. and N. Ozawa. *C^* -Algebras and Finite-dimensional Approximations*. Graduate Studies in Mathematics 88. Providence, RI: American Mathematical Society, 2008.
- [5] Bucher, M. and N. Monod. "The norm of the Euler class," preprint arXiv:1009.2316.

- [6] Burger, M. and N. Monod. "Continuous bounded cohomology and applications to rigidity theory (with an appendix by M. Burger and A. Iozzi)." *Geometric and Functional Analysis* 12, no. 2 (2002): 219–80.
- [7] Connes, A. "On the cohomology of operator algebras." *Journal of Functional Analysis* 28, no. 2 (1978): 248–53.
- [8] Diestel, J., J. H. Fourie, and J. Swart. "The Metric Theory of Tensor Products." Providence, RI: American Mathematical Society, 2008. (Grothendieck's résumé revisited)
- [9] Douglas, R. G. and P. W. Nowak. "Invariant expectations and vanishing of bounded cohomology for exact groups," (2010): preprint arXiv:1001.0718.
- [10] Dvoretzky, A. and C. A. Rogers. "Absolute and unconditional convergence in normed linear spaces." *Proceedings of the National Academy of Sciences USA* 36, no. 3 (1950): 192–7.
- [11] Gromov, M. "Volume and bounded cohomology." *Institut de Hautes Études Scientifiques. Publications Mathématiques* 1982, no. 56 (1983): 5–99.
- [12] Gromov, M. "Random walk in random groups." *Geometric and Functional Analysis* 13, no. 1 (2003): 73–146.
- [13] Grothendieck, A. "Résumé de la théorie métrique des produits tensoriels topologiques." *Sociedad Matematica Mexicana. Boletín São Paulo* 8 (1953): 1–79.
- [14] Grothendieck, A. "Produits tensoriels topologiques et espaces nucléaires." *Memoirs of the American Mathematical Society* 1955, no. 16 (1955): 140.
- [15] Haagerup, U. "All nuclear C^* -algebras are amenable." *Inventiones Mathematicae* 74, no. 2 (1983): 305–19.
- [16] Higson, N. and J. Roe. "Amenable group actions and the Novikov conjecture." *Journal für die Reine und Angewandte Mathematik* 519 (2000): 143–53.
- [17] Ivanov, N. V. "Foundations of the theory of bounded cohomology." *Journal of Soviet Mathematics* 37 (1987): 1090–115.
- [18] Johnson, B. E. *Cohomology in Banach Algebras*. Memoirs of the American Mathematical Society 127. Providence, RI: American Mathematical Society, 1972.
- [19] Kakutani, S. "Concrete representation of abstract (L) -spaces and the mean ergodic theorem." *Annals of Mathematics (2)* 42, no. 2 (1941): 523–37.
- [20] Kakutani, S. "Concrete representation of abstract (M) -spaces. (A characterization of the space of continuous functions.)." *Annals of Mathematics (2)* 42, no. 4 (1941): 994–1024.
- [21] Kirchberg, E. and S. Wassermann. "Exact groups and continuous bundles of C^* -algebras." *Mathematische Annalen* 315, no. 2 (1999): 169–203.
- [22] Lindenstrauss, J. and L. Tzafriri. *Classical Banach spaces, I and II*. Classics in Mathematics. Berlin: Springer, 1996.
- [23] Monod, N. *Continuous Bounded Cohomology of Locally Compact Groups*. Lecture Notes in Mathematics 1758. Berlin: Springer, 2001.
- [24] Monod, N. and S. Popa. "On co-amenability for groups and von Neumann algebras." *Comptes Rendus Mathématiques de l'Académie des Sciences de la Société Royale du Canada* 25, no. 3 (2003): 82–7.

- [25] Monod, N. and Y. Shalom. "Negative curvature from a cohomological viewpoint and cocycle superrigidity." *Comptes Rendus Mathématique, Académie des Sciences, Série I Mathématique, Paris* 337, no. 10 (2003): 635–8.
- [26] Ozawa, N. "Amenable actions and exactness for discrete groups." *Comptes Rendus Mathématique. Académie des Sciences, Série I Mathématique, Paris* 330, no. 8 (2000): 691–5.
- [27] Zimmer, R. J. "Amenable ergodic actions, hyperfinite factors, and Poincaré flows." *Bulletin of the American Mathematical Society* 83, no. 5 (1977): 1078–80.